NON–LINEAR HYDRODYNAMIC MODELLING OF THE IRREGULAR WIND GENERATED WATER WAVES

I. M. MARTYNOVSKY[1], A. M. SERDJUCHENKO[2]

[1]*Mykolaiv Building College of KNUCA*
[2]*Admiral Makarov National University of Shipbuilding, Mykolaiv*

The key aspects in the hydrodynamic modelling of wind-generated water waves including irregularity and non–linearity of the wave motion are presented in the report. On the first stage of the investigations the intensity and frequency structure of irregular surface wave field ware determined by using generalized 6–th parametric double-peak Hassellman frequency spectrum. Irregular wave surface ware calculated in the frame of classical spectral theory on the so-called quasi-stationary temporal intervals, but irregular wave pressure field and fluid particle velocity field have been derived as a spectral solution of the specially formulated quasi linear boundary value problem for surface waves. On the second stage local group characteristic of wave surface realizations such as amplitude envelope, phase perturbations and corresponding local values of the wave numbers and frequencies ware derived by using Hilbert transform procedure. All these values are considered as input data for the non–linear generalization of the initial quasi linear wave model on the next stages. In the procedure of the non–linear generalization of the model first of all irregular wave surface must be “saturated” by high order bounded harmonics (up to 20–th order) with special phase shift to obtain asymmetric surface profiles to be specific for wind generated waves. On the next stage fluid particle velocities on the wave surface ware calculated by using specially developed half-inverse solution of the fully non–linear boundary value problem. For the non–linear calculation of wave particle velocity and pressure fields in the fluid domain Cauchy integral formula (in 2D model) has been used. Testing and systematic numerical calculations and animated pictures displayed realistic results for irregular wave surface and hydrodynamic fields in the fluid domain.

INTRODUCTION

One of the fundamental problems in the investigation and exploration of alternative sources of energy such as solar, wind, wave, etc., is to elaborate adequate mathematical models to provide some grounding in the theory for the above mentioned sources in the nature. It is well known that ocean wave energy is one of the most intensive and powerful source of alternative energy [1], compared with other alternatives, and therefore the possibilities in the exploration of wave energy attracts the greatest interest of the investigators. Wind generated ocean waves are irregular and, in general, non–linear physical process in space and time and these features of the wave motion generate difficulties in the mathematical modelling of the ocean waves. The existing family of the wave models may be divided into the 1) hydrodynamic models, developed up to strongly non–linear waves, but only for regular and weakly irregular or group waves [2], and 2) stochastic models developed for linear and weakly non–linear irregular waves in the framework of spectral approach [3,4]. All these wave models ware used very widely in the ship and offshore hydrodynamic, oceanology etc., but one can see some shortcomings and difficulties in the the practical application of these models.

Non–linear periodic and quasi periodic hydrodynamic models can't be applied directly to the real irregular wind generated waves, but linear and weakly non–linear spectral models have purely stochastic nature.

For this reason further investigations in the modelling of ocean waves related with the elaboration of non–linear hydrodynamic models exactly for irregular water waves with realistic spectra. On this basis, hydrodynamic modelling of irregular wave surface elevation, wave pressure field and fluid particle velocity field in linear and non–linear approach for the further using in the problems of wave energy extraction is the primary goal of this work.

1 LINEAR AND QUASI LINEAR APPROACH

The most common technique in the modelling of wind generated waves, which represent the irregular surface oscillation, is a spectral approach. The main idea of this technique is that superposition of a large number of small amplitude regular harmonic waves can model moving of irregular wave surface [3]:

$$\zeta_w(x,t) = \sum_{i=1}^{N} a_i \cos(k_i x + \sigma_i t + \alpha_i),\ N \sim 10^3, \tag{1}$$

here k_i, σ_i, α_i are wave number, circular frequency and initial phase of elementary harmonics and a_i is a corresponding amplitude, determined as $a_i = \sqrt{2S(\sigma_i)\Delta\sigma_i}$, where $S(\sigma_i)$ is a frequency spectrum function and $\Delta\sigma_i$ is a small frequency interval.

To calculate these amplitudes it is important to construct the energy spectrum of wave motion depending on a given wave regime, which is usually defined by two parameters: h_s – significant wave height and T_c – average spectral period of the waves. Numerous examples of the frequency spectra of irregular wave motion in real sea conditions show that in most cases they have many local peaks [5], but the major role play the peaks (maxima) in the low–frequency energy carrying part of spectrum. According to [4], we would use two–peak spectrum as a compilation of spectra JONSWAP, which are a modification of well known Pierson–Moskowitz spectra with additional Hassellman factor γ_H.

In that model the basic partial spectrum $S_1(\sigma)$ is at lower frequencies, but the other less energy carrying spectrum $S_2(\sigma)$ – at more higher frequencies [5]. The ratio of the partial maxima of the spectra will be characterized by the value $R_S = S_{1max}(\sigma)/S_{2max}(\sigma)$, and the ratio of average spectral periods of partial spectra – by the value $R_T = T_{c2}/T_{c1}$. Four additional parameters γ_{H1}, γ_{H2}, R_S, R_T should be considered as additional features for irregular wave field that refine the energy distribution in the low-frequency range of wave spectrum. The value ratios R_S and R_T can be easily identified with the experimental spectra for the real sea conditions: $\gamma_H \approx 2\ldots5$; $R_S \approx 1.2\ldots2.0$; $R_T \approx 1.2\ldots1.5$ [5].

Thus, for the full frequency spectrum $S_w(\sigma)$ it can be written

$$\begin{aligned} &S_w(\sigma) = S_1(\sigma) + S_2(\sigma),\\ &S_i(\sigma) = S_i^{PM}(\sigma)\cdot\gamma_{Hi}^{\nu_i(\sigma)}, \quad i = 1,2. \end{aligned} \tag{2}$$

Pierson–Moskowitz spectra $S_i^{PM}(\sigma)$ in Eq. (2) determined by the well known expression [3, 4]

$$S_i^{PM}(\sigma) = 0{,}11(\tilde{h}_{si})^2\tilde{T}_{ci}(\tilde{\sigma}_i)^{-5}\exp[-0{,}44(\tilde{\sigma}_i)^{-4}]. \tag{3}$$

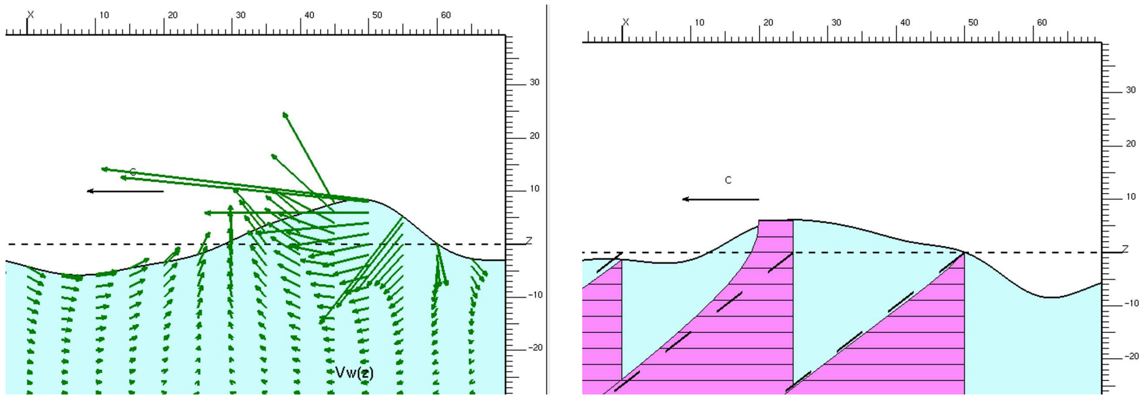


Fig. 1. Numerical modelling of hydrodynamic fields in irregular waves in the framework of classical spectral theory: the velocity field of the water particles (a) and the pressure field (b)

Here $\tilde{\sigma}_i = \sigma_i/2\pi \cdot \tilde{T}_{ci}$ is a normalized angular frequency σ for each spectrum, $\tilde{h}_{si}$ and $\tilde{T}_{ci}$ are corrected values of significant wave height and average spectral period taking into account the multiplication Hassellman factors γ_{Hi} for partial spectra.

Formally wave pressure field and particle velocity field in irregular waves may be derived in the same way as for surface elevation (see Eq. 1) $p_w(x, z, t) = \sum_{i=1}^{N} p_{wi}$, $\vec{v}_w(x, z, t) = \sum_{i=1}^{N} \vec{v}_{wi}(x, z, t)$, where p_{wi} and $\vec{v}_{wi}$ are corresponding fields for spectral components in the spectrum. But latest works [6] related to the investigation of the wave energy converters demonstrate that linear spectral model of irregular wind generated water waves to be used in the calculations have some inaccuracies in the correct calculation of the hydrodynamic pressure and velocity fields of irregular wave motion (see Fig. 1). The existence of unrealistically large values of velocities and pressures in the vicinity of wave crests related with the fact that in the linear spectral model all spectral components must run on the calm water surface, but in reality short waves run on the surface of long waves.

To eliminate these shortcomings in the calculations of hydrodynamic fields in the framework of linear spectral model special quasi linear approach in the solution of the boundary value problem for irregular water waves ware developed.

The governing equation for the two–dimensional surface wave motion in the fluid domain is the Laplace equation [3]

$$\nabla^2 \phi_w = 0,\ \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right), \tag{4}$$

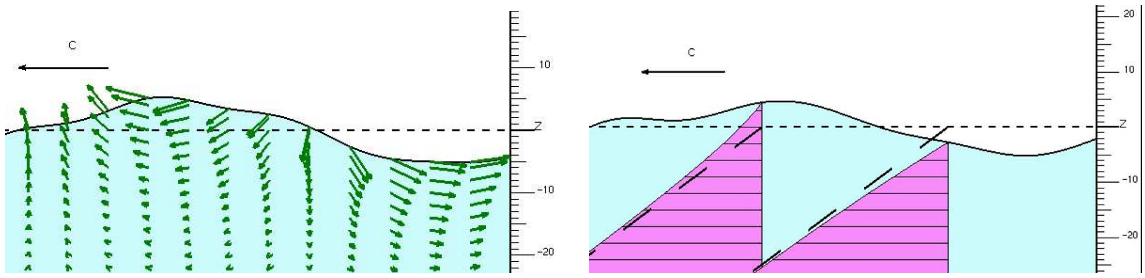
where $\phi_w(x, z, t)$ is a velocity potential as a function of spacial and temporal coordinates.

The bottom boundary condition on the depth $z = -d$ is

$$-\left.\frac{\partial \phi_w}{\partial z}\right|_{z=-d} = 0,\ d = const, \tag{5}$$

and kinematic and dynamic conditions on the moving free surface $z = \zeta_w$

$$\frac{\partial \zeta_w}{\partial t} + \frac{\partial \phi_w}{\partial x}\frac{\partial \zeta_w}{\partial x} - \frac{\partial \phi_w}{\partial z} = 0, \tag{6}$$



a *b*

Fig. 2. Numerical modelling of hydrodynamic fields in irregular waves by using corrected formulae: the velocity field of the water particles (*a*) and the pressure field (*b*)

$$\frac{\partial\phi_w}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\phi_w}{\partial x}\right)^2+\left(\frac{\partial\phi_w}{\partial z}\right)^2\right]+g\zeta_w=0, \tag{7}$$

within the Eq. (4) formulate corresponding boundary value problem for surface water waves.

In the linear approach all non–linear terms in the boundary conditions (6) and (7) must be removed from the equations and conditions must be transformed to undisturbed water level $z=0$. But we modify the linear approach and leave the linear formulation of the boundary conditions on the wave surface $z=\zeta_w$. Then the following boundary condition can be written in the form

$$\frac{\partial\zeta_w}{\partial t}=\frac{\partial\phi_w}{\partial z},\ \zeta_w=-\frac{1}{g}\frac{\partial\phi_w}{\partial t},\ z=\zeta_w. \tag{8}$$

The solution of the formulated boundary value problem can be obtained through the separation of variables technique and the resulting formula for velocity potential is as follow

$$\phi_w(x,z,t)=-\sum_{i=1}^{N}\frac{a_i g}{\sigma_i}\frac{\mathrm{ch}\,[k_i(z+d)]}{\mathrm{ch}\,[k_i(\zeta_w+d)]}\sin(k_i x+\sigma_i t+\alpha_i), \tag{9}$$

where the variables σ_i and k_i are related through the dispersion relationship in this approach

$$\sigma_i^2=gk_i\mathrm{th}\,[k_i(\zeta_w+d)]. \tag{10}$$

Then corresponding expressions for the pressure and velocity fields in irregular waves would be

$$p_w(x,z,t)=\rho g\left[-z+\sum_{i=1}^{N}a_i\frac{\mathrm{ch}\,[k_i(z+d)]}{\mathrm{ch}\,[k_i(\zeta_w+d)]}\cos(k_i x+\sigma_i t+\alpha_i)\right], \tag{11}$$

$$\upsilon_{\{x,z\}}(x,z,t)=-\sum_{i=1}^{N}a_i\sigma_i\frac{\{\mathrm{ch}\,,\mathrm{sh}\,\}[k_i(z+d)]}{\mathrm{sh}\,[k_i(\zeta_w+d)]}\{\cos,\sin\}(k_i x+\sigma_i t+\alpha_i). \tag{12}$$

Typical sample of the calculations of the pressure and velocity fields by using Eq. (11) and (12) is displayed in Fig. 2 and it is very clear that pressure and velocity in the vicinity of wave surface are looking quite realistic without extensive fluctuations.

For the further fully non–linear generalization of the quasi linear solution we have to derive local characteristics of irregular wave motion: amplitude envelope, phase modulations,

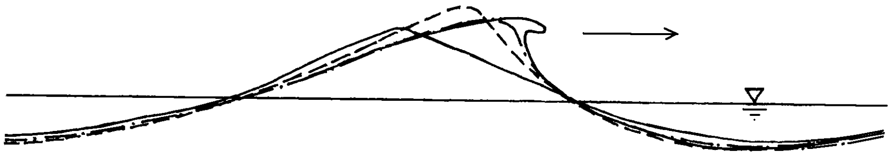


Fig. 3. Differences between linear and non–linear water wave profile

wave steepness etc. For every fixed time t_k these values may be derived by using well known Hilbert transform of the wave profile $\zeta_w(x,t_k)$. The resulting formulae are as follow

$$\left.\begin{array}{lll} a_w(x,t_k)=\sqrt{\zeta_w^2+\xi_w^2}, & \theta(x,t_k)=\arctan\left(-\xi_w/\zeta_w\right), & \delta_w(x,t_k)=a_w\cdot k_w, \\ k_w(x,t_k)=\partial\theta/\partial x, & \xi_w(x,t)=\frac{1}{\pi}\nu.\rho.\int_{-\infty}^{\infty}\frac{\zeta_w(x,x')}{x'-x}\,dx' & \end{array}\right\} \quad (13)$$

and wave surface may be rewritten in the form

$$\zeta_w(x,t)=a_w(x,t)\cos\theta(x,t), \quad (14)$$

where $\theta=\theta_0+\psi(x,t)$; $\theta_0=\langle k\rangle x+\langle\sigma\rangle t+\alpha_0$ – average phase and $\psi(x,t)$ – perturbation phase.

2 FULLY NON–LINEAR APPROACH

Developed in this part of the report fully non–linear approach is an approximate approach, based on some assumptions and results derived in the previous investigations of non–linear periodic and modulated wave motions. First of all differences between linear and non–linear water wave profile are shown in Fig. 3, and specific non–linear wave profile can be described by adding into the Eq. (14) some numbers of high order bounded harmonics $a_n\cos n\theta_n$ with special phase shift to obtain asymmetric surface profiles to be specific for wind generated waves. Then we obtain the expression

$$\zeta_w(x,t)\cong\sum_{n=1}^{M}a_n\cos(\langle k\rangle(x+ct)n+\alpha_n+n\psi),\ M\sim 20-25, \quad (15)$$

where $a_1\equiv a$, $a_n=f(a,\psi)$ and $c=\langle c\rangle\cdot\nu(\delta_w)=\langle c\rangle(1+1/2\delta_w^2+1/8\delta_w^8)$ local phase velocity involving non–linear correction.

In this approach wave elevation and its derivatives in the boundary conditions (6) and (7) may be considered as known values. If we would consider surface velocities v_{wx}^s and v_{wz}^s as main unknown values in these equations, then the general problem associated with the implementation of above mentioned technique is how replace temporal derivative velocity for potential $(\partial\phi_w/\partial t)^s\rightarrow(\partial\phi_w/\partial x)\equiv v_{wx}^s$ and in general for the free surface elevation $\partial\zeta_w/\partial t\rightarrow\partial\zeta_w/\partial x$ in the boundary conditions. For the beginning write down the follow expressions for the temporal derivatives

$$\frac{\partial\phi_w}{\partial t}=c_\phi\frac{\partial\phi_w}{\partial x},\ \frac{\partial\zeta_w}{\partial t}=c_\zeta\frac{\partial\zeta_w}{\partial x}, \quad (16)$$

where c_ϕ and c_ζ in general are some operators which have the physical sense of local phase velocities.

In the framework of linear spectral theory these phase velocities may be written in the form

$$c_\phi \cong \frac{\sum\limits_{j=1}^{N} a_j \cos\theta_j}{\sum\limits_{j=1}^{N} a_j \dfrac{k_j}{\sigma_j} \cos\theta_j},\ c_\zeta \cong \frac{\sum\limits_{j=1}^{N} a_j \sigma_j \sin\theta_j}{\sum\limits_{j=1}^{N} k_j \sin\theta_j}, \tag{17}$$

but it is a compromise solution now and additional investigations must be payed for the more correct non–linear estimations of c_ϕ and c_ζ for fully irregular wave motions. Then equation (6) and (7) can be rewritten as algebraic equation for υ^s_{wx}, υ^s_{wz} $c_\zeta\zeta_x + \zeta_x\upsilon^s_{wx} - \upsilon^s_{wz} = 0, c_\phi\upsilon^s_{wx} + \frac{1}{2}((\upsilon^s_{wx})^2 + (\upsilon^s_{wz})^2) - g\zeta_w = 0$. The solution of these equations are

$$\begin{aligned} \upsilon^s_{wx} &= -c_\zeta \left[\left(1 - \frac{\sqrt{P_- - \delta P_-}}{\sqrt{P_+}} \right) + \zeta_x \cdot \delta P_\gamma \right], \\ \upsilon^s_{wz} &= -c_\zeta \left[\zeta_x \frac{\sqrt{P_- - \delta P_-}}{\sqrt{P_+}} + \delta P_\gamma \right], \end{aligned} \tag{18}$$

where it is denoted $P_+ = (1 + \zeta_x^2)$, $P_- = (1 - 2g\zeta/c_\zeta^2)$, $\delta P_- = \zeta_x \cdot \delta P_\gamma(\gamma - 1)$, $\delta P_\gamma = \zeta_x(\gamma - 1)/(1 + \zeta_x^2)$, $\gamma = c_\phi/c_\zeta$.

Comparison of the results obtained by using Eq. (18) of experimental investigations [7] and with the results of the numerical solution of the fully non–linear problem by using Boundary Elements Method [8] displayed quite good coincidence of surface velocities.

Further, in the approaching of 2D wave problem projections velocity υ^s_{wx}, υ^s_{wz} can be considered as real and imaginary parts of the complex velocity on the wave surface $\tilde{\upsilon}^s = \upsilon^s_{wx} - i\upsilon^s_{wz}$ and then the velocity field in the fluid domain can be obtained using the technique of Cauchy integrals [7]

$$\tilde{\upsilon}_w(\tilde{z}) = \frac{-i}{2\pi} \int\limits_{(z=\zeta_w)} \frac{\tilde{\upsilon}^s(\tilde{\zeta})}{\tilde{\zeta} - \tilde{z}}\, d\tilde{\zeta}, \quad \tilde{z} = x + iz, \quad \tilde{\zeta} \in \zeta_w, \tag{19}$$

here $\tilde{z}$ – fixed point in the water, $\tilde{\zeta}$ – the current point on the wave surface.

The field of hydrodynamic pressure in waves is determined by a well–known Cauchy–Lagrange equation $p_w = -\rho[\Phi_{wt} + \frac{1}{2}(\nabla\Phi_w \cdot \nabla\Phi_w) + gz]$, which, after replacing the variables can be rewritten as $p_w \approx -\rho[c_\phi\upsilon_{wx} + \frac{1}{2}(\upsilon^2_{wx} + \upsilon^2_{wz}) + gz]$. Equation (15)÷(19) may be used for non–linear estimation of wave characteristics of the initial time t_0. For the next time step $t_0 + \Delta t$ the wave elevation can be determined by using numerical integration of the equation $\partial\zeta_w/\partial t = -\partial/\partial x \int_{-d}^{\zeta_w} \upsilon_{wx}\, dz$, which have been obtained from the integration of continuity equation (4) in z direction and by using the kinematic boundary conditions on the wave surface. Further calculations must be repeated similar to the first time step as it was described above.

CONCLUSIONS

A quasi linear and non–linear theory in the modelling of fully irregular wind generated waves, including wave pressure field and fluid particle velocity field calculations are obtained. The results can be used in further investigations of the wave energy converters, determining their efficiency, optimal configuration schemes etc.

REFERENCES

1. *Khan J., Bhuyan G. S., Moshref A.* Potential Opportunities and Differences Associated with Integration of Ocean Wave and Marine Current Energy Plants in Comparison to Wind Enery // Final Annex III Technical Report.– 2009.– IEA-OES Document No T0311.– P. 64.

2. *Сердюченко А. М.* Нелінійні задачі гідродинаміки вітрових хвиль та суден в умова морського хвилювання // Дисертація на здобуття наукового ступеня д.ф.–м.н.– 2004, Інститут гідромеханіки НАНУ.– С. 297.

3. *Луговский В. В.* Динамика моря .– Л.: Судостроение, 1976.– 201 с.

4. *Ochi M. K.* Ocean Waves: The Stochastic Approach.– USA: Cambridge University Press, 2005.– 319 p.

5. *Soares C. G.* Representation of double-peaked sea wave spectra // Ocean Engineering.– 1984.– N 11.– P. 185–207.

6. *Мартиновськиий І. М., Сердюченко А. Н.* Чисельне модулювання нерегулярних вітрових хвиль в задачах хвильової енергетики // Вісник ХНТУ.– 2011.– N 3(**42**).– С. 290–295.

7. *Melville W. K., Rapp R. J.* The surface velocity field in steep and breaking waves // J. Fluid Mech.– 1988.– Vol.**188**.– P. 1-22.

8. *Takagi K., Naito S., Nakamura S.* Computation of Nonlinear Hydrodynamic Forces on Two-Dimensional Body by Boundary Element Method // J.Kansai Soc. Naval Arch., Japan.– 1985.– N 197.– P. 31-38.